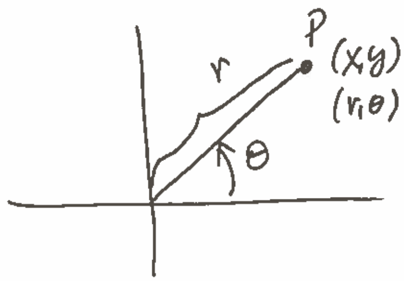
15.7 Triple Integrals in cylindrical coords.

Extension of polar coords. to 3-D.

Polar:

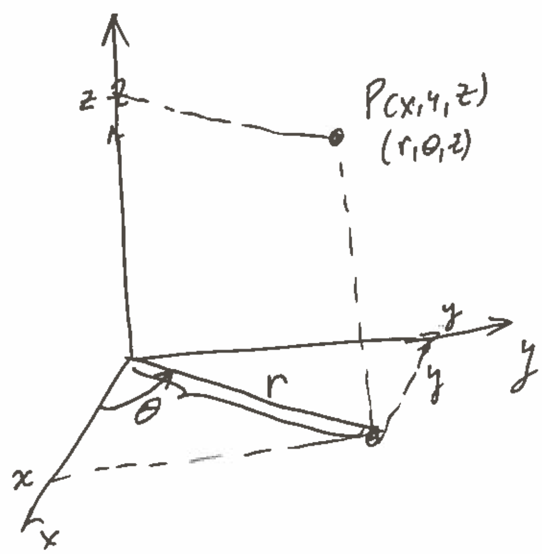


$(x,y) \longleftrightarrow (r,\theta)$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ direct transf

$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan \frac{y}{x} \end{cases}$ Inverse transf

Cylindrical



$(x,y,z) \Longleftrightarrow (r,\theta,z)$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

Triple Integrals in Cylindrical coords.

Consider a type I region $E = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

$$I = \iiint_E f\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

Representing D in polar coords.

$$\Rightarrow \quad I = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

15.8 Triple Integrals in Cylindrical Coordinates.

Motivation

Find the Volume of Ice-Cream Cone

bounded below by the upper half Cone: $z^2 = 3(x^2+y^2)$

$$z = \sqrt{3(x^2+y^2)}$$

And above by the unit sphere:

$$x^2+y^2+z^2 = 1$$

$$z^2 = 3x^2+3y^2$$

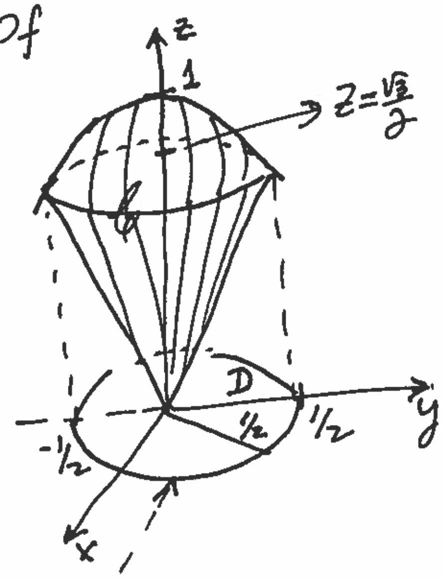
First,

Find the curve of intersection $\mathcal{C}$ of

Cone: $z^2 = 3x^2 + 3y^2$ (1)

and

Sphere: $x^2 + y^2 + z^2 = 1$ (2)



$z^2 = 1 - x^2 - y^2$

$\Rightarrow \boxed{3x^2 + 3y^2 = 1 - x^2 - y^2}$ gives the curve of intersection $\mathcal{C}$ projected onto the xy-plane

$\Rightarrow 4x^2 + 4y^2 = 1 \Rightarrow x^2 + y^2 = \frac{1}{4}$ (3) Circle of radius $\frac{1}{2}$

The equations representing the actual curve of intersection are

$$\mathcal{C}: \begin{cases} x^2 + y^2 = \frac{1}{4} \\ z = \frac{\sqrt{3}}{2} \end{cases}$$

Since replacing (3) into (2)

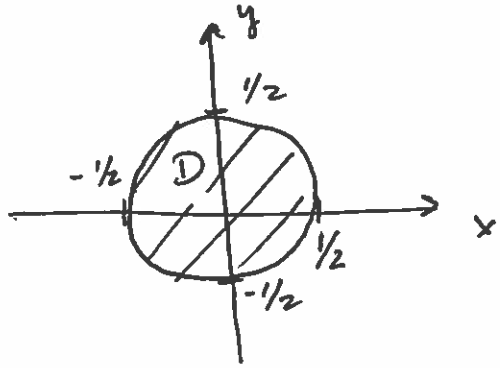
$z^2 = 1 - x^2 - y^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow z = \frac{\sqrt{3}}{2}$.

From these two equs., we can easily get 3 parametric equs. using polar coordinates.

$$\mathcal{C}: \begin{cases} x = \frac{1}{2}\cos\theta, & 0 \le \theta \le 2\pi \\ y = \frac{1}{2}\sin\theta \\ z = \frac{\sqrt{3}}{2} \end{cases}$$

The domain D of integration is



Therefore,

$$\text{Vol } E = \iint\limits_D \Bigg(\underset{\substack{\text{and Cylinder}\\ x^2+y^2=1/4}}{\overset{\text{Vol enclosed by upper}}{\text{Sphere}}} - \underset{\substack{\text{and}\\ \text{above xy-plane}}}{\overset{\text{Vol below upper}}{\text{cone}}} \Bigg)\, dA$$

Cartesian

$$= \int_{x=-1/2}^{1/2} \int_{y=-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}} \left(\sqrt{1-x^2-y^2} - \sqrt{3(x^2+y^2)} \right) \underbrace{dy\,dx}_{dA}$$

$$= \underbrace{\int_{x=-1/2}^{1/2} \int_{y=-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}}}_{D} \int_{\underset{\substack{\text{(Surface below)}\\ \text{Cone}}}{\sqrt{3(x^2+y^2)}}}^{\overset{\substack{\text{Sphere}\\ \text{(Surface on top)}}}{\sqrt{1-x^2-y^2}}} \underbrace{dz\,dy\,dx}_{dV} =$$

Introducing polar Coordinates for the domain D.

$x = r\cos\theta, \quad y = r\sin\theta,$ and keeping $z = z$.

$$I = \int_0^{2\pi}\int_0^{1/2} \left(\sqrt{1-r^2} - \sqrt{3r^2}\right) r\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{1/2} \left(r\sqrt{1-r^2} - \sqrt{3}\,r^2\right) dr\,d\theta =$$

$$= \int_0^{2\pi}\left(\frac{-(1-r^2)^{3/2}}{3} - \frac{\sqrt{3}}{3}r^3\right)\Bigg|_0^{1/2} d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[(1-r^2)^{3/2} + \sqrt{3}\,r^3\right]_0^{1/2} d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[\left(\frac{3}{4}\right)^{3/2} + \frac{\sqrt{3}}{8} - 1\right] d\theta =$$

$$= -\frac{1}{3}\int_0^{2\pi}\left(\frac{3\sqrt{3}}{8} + \frac{\sqrt{3}}{8} - 1\right) d\theta = -\frac{1}{3}\int_0^{2\pi}\frac{\sqrt{3}-2}{2}\,d\theta$$

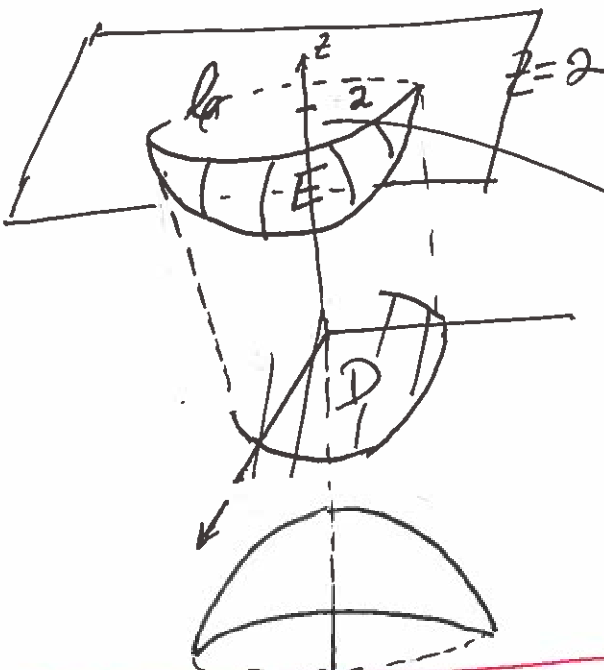
$$= -\frac{1}{3}(\cancel{2}\pi)\left(\frac{\sqrt{3}-2}{\cancel{2}}\right) = \frac{2-\sqrt{3}}{3}\pi$$

$$u = 1-r^2$$
$$du = -2r\,dr$$
$$\int r(1-r^2)^{1/2}\,dr =$$
$$= -\frac{1}{2}\int (1-r^2)^{1/2}(-2r)\,dr$$
$$= -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{2}\frac{u^{3/2}}{3/2}$$
$$= -\frac{u^{3/2}}{3} = -\frac{(1-r^2)^{3/2}}{3}$$

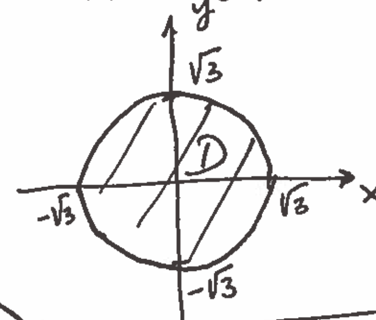
Exercise: Compute the Volumen of Solid

enclosed by

$$\begin{cases} \text{Hyperboloid: } & x^2+y^2-z^2=-1 \quad \text{and} \\ \text{plane: } & z=2. \end{cases}$$



Projection of Region E into xy-plane.



Curve of Intersection two Surfaces: $x^2+y^2-(2)^2=-1$

$$\boxed{x^2+y^2=3}$$

$$\boxed{z=2}$$

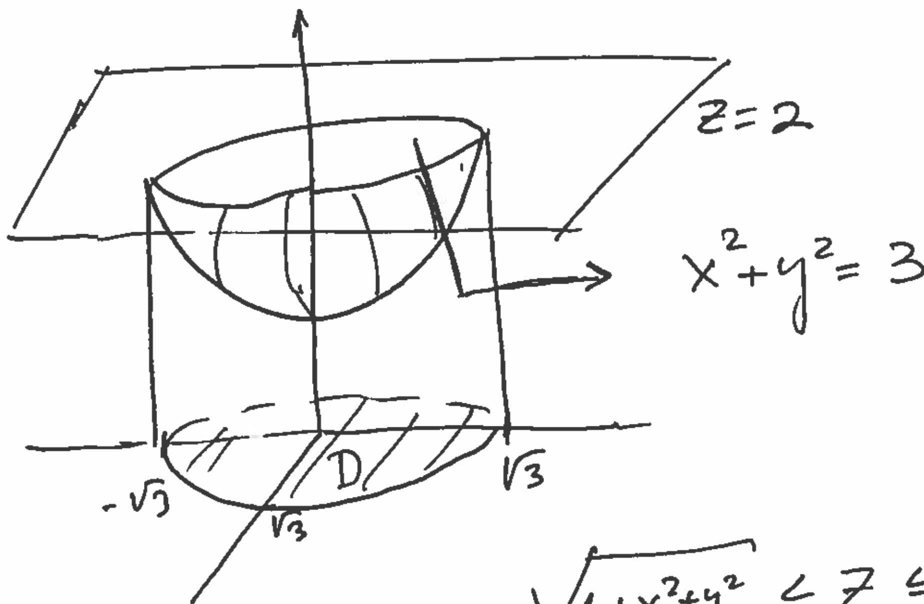
This is a Type I region E in space.

Then,

$$Vol(E)=\iiint_E dv$$

or

$$Vol(E)=\iint_D \int_{z=\sqrt{x^2+y^2+1}}^{z=2} dz\, dA$$



$$\sqrt{1+x^2+y^2} \le z \le 2$$

$$\left\{\begin{array}{ll} \text{Hyperboloid:} & -x^2-y^2+z^2=1 \\ \text{plane:} & z=2 \end{array}\right\}$$

So,

$$\mathrm{Vol}(E) = \iiint_E dv$$

or

$$\mathrm{Vol}(E) = \iint_D \int_{z=\sqrt{x^2+y^2+1}}^{z=2} dv$$

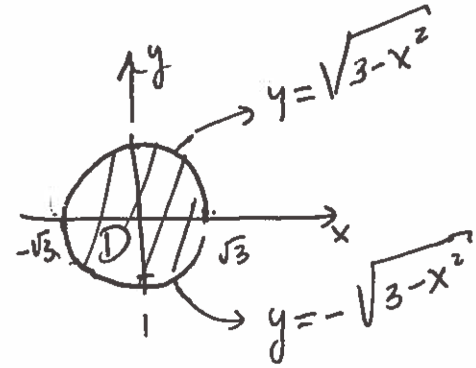
(Top Surface) $z=2$

$z=\sqrt{x^2+y^2+1}$ (Surface below)

In Cartesian Coords.

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{\sqrt{x^2+y^2+1}}^{2} dz\,dy\,dx$$

$x \quad y \quad z$



In polar coords. for x and y

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{r^2+1}}^{2} dz\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2-\sqrt{r^2+1}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2r - r\sqrt{r^2+1}\right) dr\,d\theta = \int_0^{2\pi} r^2\Big|_0^{\sqrt{3}} d\theta - \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{3}} \underbrace{\sqrt{r^2+1}}_{=(r^2+1)^{1/2}}\, 2\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[(\sqrt{3})^2 - \frac{1}{\cancel{2}}(r^2+1)^{3/2}\cdot\frac{\cancel{2}}{3}\Big|_0^{\sqrt{3}}\right] d\theta = \int_0^{2\pi}\left[3 - \left(\frac{1}{3}(3+1)^{3/2} - \frac{1}{3}\right)\right] d\theta$$

$$= 2\pi\left(3 - \frac{1}{3}(8) + \frac{1}{3}\right) = \frac{4\pi}{3}$$